

# THE CONFIGURATION SPACE OF EQUIDISTANT TRIPLES IN THE HEISENBERG GROUP

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**ABSTRACT.** We prove that the configuration space of equidistant triples on the Heisenberg group equipped with the Korányi metric, is isomorphic to a hypersurface of  $\mathbb{R}^3$ .

## 1. INTRODUCTION

Let  $\mathfrak{H}$  be the first Heisenberg group equipped with the Korányi distance  $d$ . An equidistant triple is a triple of points  $P = (p_1, p_2, p_3) \in \mathfrak{H}$  such that

$$d(p_1, p_2) = d(p_2, p_3) = d(p_3, p_1).$$

Denote by  $\mathcal{ET}$  the space of equidistant triples in  $\mathfrak{H}$ . Then the similarity group

$$G = \text{Sim}(\mathfrak{H}, d) = \mathfrak{H} \times \mathbb{R} \times \mathbb{R}_+^*,$$

acts diagonally on  $\mathcal{ET}$ :

$$(g, (p_1, p_2, p_3)) \mapsto (g(p_1), g(p_2), g(p_3)).$$

We denote by  $\mathfrak{ET}$  the quotient of this action; this is the *configuration space of  $G$ -equivalent equidistant triples in  $\mathfrak{H}$* . In this paper we are dealing with the problem of parametrising  $\mathfrak{ET}$ . The problem is addressed and solved in a different manner in [1] (see Proposition 4.6 there). We prove here the following theorem:

**Theorem 1.1.** *The configuration space  $\mathfrak{ET}$  of  $G$ -equivalent equidistant triples is in bijection with the hypersurface  $\mathcal{E}$  of  $\mathbb{R}^3$  which is defined by*

$$\mathcal{E} = \left\{ (a, b, c) \in [-2\pi/3, 2\pi/3]^3 \mid \cos a + \cos b + \cos c = \frac{3}{2} \right\}.$$

The proof of Theorem 1.1 relies upon the use of the cross-ratio variety  $\mathfrak{X}$ ; this is a 4-dimensional variety parametrising the  $\text{PU}(2, 1)$ -configuration space of pairwise distinct quadruples on the boundary  $\partial\mathbf{H}_{\mathbb{C}}^2$  of complex hyperbolic plane. In fact,  $\mathfrak{ET}$  may be viewed as a 2-dimensional subvariety of  $\mathfrak{X}$ .

This paper is organised as follows: In Section 2 we review standard facts about complex hyperbolic plane and the Heisenberg group, as well as about cross-ratio variety. In Section 3 we prove Theorem 1.1 and discuss the particular case of equidistant triples lying in a  $\mathbb{C}$ -circle in Section 3.4.

*Acknowledgements.* I wish to thank Vassilis Chousionis for suggesting the problem to me, and also Viktor Schroeder for fruitful discussions.

*Date:* March 29, 2017

2010 *Mathematics Subject Classifications.* 32M99, 51F99.

*Key words.* Heisenberg group, Korányi metric, equidistant triples.

## 2. PRELIMINARIES

The material of this section is standard; a general reference is Goldman's book, [5]. In Section 2.1 we review complex hyperbolic plane, its boundary and the Heisenberg group. Cartan's angular invariant and complex cross-ratios are in Section 2.2. Finally, a brief overview of the cross-ratio variety and the  $\mathrm{PU}(2, 1)$ -configuration of four pairwise distinct points in the boundary of complex hyperbolic plane is found in Section 2.3.

**2.1. Complex hyperbolic plane and Heisenberg group.** We consider the vector space  $\mathbb{C}^{2,1}$ , that is,  $\mathbb{C}^3$  with the Hermitian form of signature  $(2, 1)$  given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1.$$

We next consider the following subspaces of  $\mathbb{C}^{2,1}$ :

$$V_- = \left\{ \mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0 \right\}, \quad V_0 = \left\{ \mathbf{z} \in \mathbb{C}^{2,1} \setminus \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0 \right\}.$$

Denote by  $\mathbb{P} : \mathbb{C}^{2,1} \setminus \{0\} \rightarrow \mathbb{CP}^2$  the canonical projection onto complex projective space. Then the *complex hyperbolic plane*  $\mathbf{H}_{\mathbb{C}}^2$  is defined to be  $\mathbb{P}V_-$  and its boundary  $\partial \mathbf{H}_{\mathbb{C}}^2$  is  $\mathbb{P}V_0$ . Hence we have

$$\mathbf{H}_{\mathbb{C}}^2 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_1) + |z_2|^2 < 0 \right\},$$

and in this manner,  $\mathbf{H}_{\mathbb{C}}^2$  is the Siegel domain in  $\mathbb{C}^2$ .

There are two distinguished points in  $V_0$  which we denote by  $\mathbf{o}$  and  $\infty$ :

$$\mathbf{o} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Let  $\mathbb{P}\mathbf{o} = o$  and  $\mathbb{P}\infty = \infty$ . Then

$$\partial \mathbf{H}_{\mathbb{C}}^2 \setminus \{\infty\} = \left\{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_1) + |z_2|^2 = 0 \right\},$$

and in particular,  $o = (0, 0) \in \mathbb{C}^2$ .

Conversely, if we are given a point  $z = (z_1, z_2)$  of  $\mathbb{C}^2$ , then the point

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}.$$

is called the *standard lift* of  $z$ . Therefore the standard lifts of points of the complex hyperbolic plane and its boundary (except the point at infinity) are vectors of  $V_-$  and  $V_0$  respectively with the third inhomogeneous coordinate equal to 1.

Complex hyperbolic plane  $\mathbf{H}_{\mathbb{C}}^2$  is a Kähler manifold; its Kähler structure is given by the Bergman metric. The holomorphic sectional curvature equals to  $-1$  and its real sectional curvature is pinched between  $-1$  and  $-1/4$ . The full group of holomorphic isometries is the *projective unitary group*

$$\mathrm{PU}(2, 1) = \mathrm{SU}(2, 1) / \{I, \omega I, \omega^2 I\},$$

where  $\omega$  is a non-real cube root of unity (that is  $\mathrm{SU}(2, 1)$  is a 3-fold covering of  $\mathrm{PU}(2, 1)$ ). There are two ways (up to  $\mathrm{PU}(2, 1)$  conjugacy) to embed real hyperbolic plane into complex hyperbolic plane; that is, as  $\mathbf{H}_{\mathbb{C}}^1$  as well as  $\mathbf{H}_{\mathbb{R}}^2$ . These embeddings give rise to complex lines,

i.e., isometric images of the embedding of  $\mathbf{H}_{\mathbb{C}}^1$  into  $\mathbf{H}_{\mathbb{C}}^2$  and Lagrangian planes, i.e., isometric images of  $\mathbf{H}_{\mathbb{R}}^2$  into  $\mathbf{H}_{\mathbb{C}}^2$ , respectively.

There is an identification of the boundary of the Siegel domain with the one point compactification of  $\mathbb{C} \times \mathbb{R}$ : A finite point  $z$  in the boundary of the Siegel domain has a standard lift of the form

$$\mathbf{z} = \begin{bmatrix} -|z|^2 + it \\ \sqrt{2}z \\ 1 \end{bmatrix}.$$

The unipotent stabiliser at infinity acts simply transitively and gives the set of these points the structure of a 2-step nilpotent Lie group, namely the Heisenberg group  $\mathfrak{H}$ . This is  $\mathbb{C} \times \mathbb{R}$  with group law:

$$(z, t) \star (w, s) = (z + w, t + s + 2\Im(z\bar{w})).$$

The Heisenberg norm (Korányi gauge) is given by

$$|(z, t)|_{\mathfrak{H}} = |\mathcal{A}(z, t)|^{1/2}, \quad \text{where } \mathcal{A}(z, t) = |z|^2 - it.$$

From this norm arises a metric, the Korányi-Cygan (K-C) metric, on  $\mathfrak{H}$  by the relation

$$d((z, t), (w, s)) = |(z, t)^{-1} \star (w, s)|_{\mathfrak{H}}.$$

The K-C metric is invariant under

- a) left-actions  $L_{(w, s)}$  of  $\mathfrak{H}$ ,  $(z, t) \rightarrow (w, s) \star (z, t)$ ,  $(w, s) \in \mathfrak{H}$ ;
- b) rotations  $R_{\phi}$ ,  $(z, t) \mapsto (ze^{i\phi}, t)$ ,  $\phi \in \mathbb{R}$ ;
- c) involution  $j$ ,  $j(z, t) = (\bar{z}, -t)$ .

These form the group  $\text{Isom}(\mathfrak{H}, d)$  of *Heisenberg isometries*. Note that all the above are orientation-preserving.

The K-C metric is also scaled up to multiplicative constants by the action of

- d) Heisenberg dilations  $D_r$ ,  $(z, t) \mapsto (rz, r^2t)$ ,  $r \in \mathbb{R}_+^*$ .

and there is also an inversion, defined for each  $p = (z, t) \in \mathfrak{H}$ ,  $p \neq o$ , by

$$(z, t) \mapsto \left( \frac{z}{-|z|^2 + it}, -\frac{t}{|-|z|^2 + it|^2} \right),$$

which satisfies

$$d_{\mathfrak{H}}(R(p), R(p')) = \frac{d_{\mathfrak{H}}(p, p')}{d_{\mathfrak{H}}(p, o)d_{\mathfrak{H}}(p', o)}.$$

The similarity group  $G = \text{Sim}(\mathfrak{H}, d)$  comprises compositions of maps of the form a), b), d). Clearly,  $G = \mathfrak{H} \times \mathbb{R} \times \mathbb{R}_+^*$ .

**2.1.1.  $\mathbb{R}$ -circles and  $\mathbb{C}$ -circles.**  $\mathbb{R}$ -circles are boundaries of Lagrangian planes and  $\mathbb{C}$ -circles are boundaries of complex lines. They come in two flavours, infinite ones (i.e., containing the point at infinity) and finite ones. We refer to [5] for more details about these curves.

**2.2. Cartan's Angular Invariant.** Given a triple  $(p_1, p_2, p_3)$  of points at the boundary  $\partial\mathbf{H}_{\mathbb{C}}^2$  the Cartan's angular invariant  $\mathbb{A}(p_1, p_2, p_3)$  is defined by

$$\mathbb{A}(p_1, p_2, p_3) = \arg(-\langle \mathbf{p}_1, \mathbf{p}_2 \rangle \langle \mathbf{p}_2, \mathbf{p}_3 \rangle \langle \mathbf{p}_3, \mathbf{p}_1 \rangle),$$

where  $\mathbf{p}_i$  are lifts of  $p_i$ ,  $i = 1, 2, 3$ . The Cartan's angular invariant lies in  $[-\pi/2, \pi/2]$ , is independent of the choice of the lifts and remains invariant under the diagonal action of  $\mathrm{PU}(2, 1)$ . Any other permutation of points produces angular invariants which differ from the above possibly up to sign. The following propositions are in [5] to which we also refer the reader for further details:

**Proposition 2.1.** *Let  $(p_1, p_2, p_3)$  be a triple of points lying in  $\partial\mathbf{H}_{\mathbb{C}}^2$  and let also  $\mathbb{A} = \mathbb{A}(p_1, p_2, p_3)$  be their Cartan's angular invariant. Then:*

- (1) *All points lie in an  $\mathbb{R}$ -circle if and only if  $\mathbb{A} = 0$ .*
- (2) *All points lie in a  $\mathbb{C}$ -circle if and only if  $\mathbb{A} = \pm\pi/2$ .*

**Proposition 2.2.** *Suppose that  $p_i$  and  $p'_i$ ,  $i = 1, 2, 3$ , are points in  $\partial\mathbf{H}_{\mathbb{C}}^2$ . If there exists a holomorphic isometry  $g$  of  $\mathbf{H}_{\mathbb{C}}^2$  such that  $g(p_i) = p'_i$ ,  $i = 1, 2, 3$ , then  $\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p'_1, p'_2, p'_3)$ . Conversely, if  $\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p'_1, p'_2, p'_3)$ , then there exists a holomorphic isometry  $g$  of  $\mathbf{H}_{\mathbb{C}}^2$  such that  $g(p_i) = p'_i$ ,  $i = 1, 2, 3$ . This isometry is unique unless  $p_i$ ,  $i = 1, 2, 3$ , lie in a  $\mathbb{C}$ -circle.*

**2.3. Cross-ratio variety and the configuration space.** Given a quadruple of pairwise distinct points  $\mathbf{p} = (p_1, p_2, p_3, p_4)$  in  $\partial\mathbf{H}_{\mathbb{C}}^2$ , we define their complex cross-ratio as follows:

$$\mathbb{X}(p_1, p_2, p_3, p_4) = \frac{\langle \mathbf{p}_3, \mathbf{p}_1 \rangle \langle \mathbf{p}_4, \mathbf{p}_2 \rangle}{\langle \mathbf{p}_4, \mathbf{p}_1 \rangle \langle \mathbf{p}_3, \mathbf{p}_2 \rangle},$$

where  $\mathbf{p}_i$  are lifts of  $p_i$ ,  $i = 1, 2, 3, 4$ , see also [6], [7], [8]. The cross-ratio is independent of the choice of lifts and remains invariant under the diagonal action of  $\mathrm{PU}(2, 1)$ . We stress here that for points in the Heisenberg group, the square root of its absolute value is

$$|\mathbb{X}(p_1, p_2, p_3, p_4)|^{1/2} = \frac{d_{\mathfrak{H}}(p_4, p_2) \cdot d_{\mathfrak{H}}(p_3, p_1)}{d_{\mathfrak{H}}(p_4, p_1) \cdot d_{\mathfrak{H}}(p_3, p_2)}.$$

Given a quadruple  $\mathbf{p} = (p_1, p_2, p_3, p_4)$  of pairwise distinct points in the boundary  $\partial\mathbf{H}_{\mathbb{C}}^2$ , all possible permutations of points gives us 24 complex cross-ratios corresponding to  $\mathbf{p}$ . Due to symmetries, see [3], Falbel showed that all cross-ratios corresponding to a quadruple of points depend on three cross-ratios which satisfy two real equations. Indeed, the following proposition holds; for its proof, see for instance [7].

**Proposition 2.3.** *Let  $\mathbf{p} = (p_1, p_2, p_3, p_4)$  be any quadruple of pairwise distinct points in  $\partial\mathbf{H}_{\mathbb{C}}^2$ . Let*

$$\mathbb{X}_1(\mathbf{p}) = \mathbb{X}(p_1, p_2, p_3, p_4), \quad \mathbb{X}_2(\mathbf{p}) = \mathbb{X}(p_1, p_3, p_2, p_4), \quad \mathbb{X}_3(\mathbf{p}) = \mathbb{X}(p_2, p_3, p_1, p_4).$$

*Then*

$$(2.1) \quad |\mathbb{X}_3|^2 = |\mathbb{X}_2|^2 / |\mathbb{X}_1|^2,$$

$$(2.2) \quad 2|\mathbb{X}_1|^2 \Re(\mathbb{X}_3) = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2\Re(\mathbb{X}_1) - 2\Re(\mathbb{X}_2) + 1.$$

Equations (2.1) and (2.2) define a 4-dimensional real subvariety of  $\mathbb{C}^3$  which we call the *cross-ratio variety*  $\mathfrak{X}$ . This variety is isomorphic to the subset  $\mathfrak{F}'$  of the  $\mathrm{PU}(2, 1)$  configuration space  $\mathfrak{F}$  of pairwise disjoint quadruples of points in  $\partial\mathbf{H}_{\mathbb{C}}^2$ , comprising quadruples whose points do not all lie in the same  $\mathbb{C}$ -circle. In the latter case, we have a 2–1 map between the subset  $\mathfrak{F}_{\mathbb{R}}$  comprising of quadruples whose points all lie in a  $\mathbb{C}$ -circle and the subvariety  $\mathfrak{X}_{\mathbb{R}}$  of  $\mathfrak{X}$  defined by

$$\mathfrak{X}_{\mathbb{R}} = \{(\mathbb{X}_1, \mathbb{X}_2) \in \mathbb{R}_*^2 \mid \mathbb{X}_1 + \mathbb{X}_2 = 1\},$$

see [3], [2]. For further reference, we shall need the following:

**Remark 2.4.** Let  $\mathbf{p}$  a quadruple of pairwise distinct points in  $\partial\mathbf{H}_{\mathbb{C}}^2$  which do not all lie in the same  $\mathbb{C}$ -circle and let also  $\mathbb{X}_i(\mathbf{p})$ ,  $i = 1, 2, 3$  be as above. Let also  $a = \arg(\mathbb{X}_1)$ ,  $b = \arg(\mathbb{X}_2)$ ,  $c = \arg(\mathbb{X}_3)$ . We have

$$a = \mathbb{A}_1 - \mathbb{A}_2, \quad b = -\mathbb{A}_2 - \mathbb{A}_4, \quad c = \mathbb{A}_4 - \mathbb{A}_1.$$

Here,

$$\mathbb{A}_1 = \mathbb{A}(p_2, p_3, p_4), \quad \mathbb{A}_2 = \mathbb{A}(p_1, p_3, p_4), \quad \mathbb{A}_4 = \mathbb{A}(p_1, p_2, p_3).$$

As for  $\mathbb{A}_3 = \mathbb{A}(p_1, p_2, p_4)$  we have

$$\mathbb{A}_3 + \mathbb{A}_1 = \mathbb{A}_2 + \mathbb{A}_4.$$

### 3. THE CONFIGURATION SPACE OF EQUIDISTANT TRIPLES

In this section we are going to prove Theorem 1.1. The proof will follow after a series of lemmas which follow below.

**3.1. The lemmas.** Throughout this section we will have the following notation: We shall denote by  $P$  a triple  $(p_1, p_2, p_3)$  of pairwise distinct points in the Heisenberg group  $\mathfrak{H}$ . We will consider also the quadruple  $\mathbf{p} = (p_1, \infty, p_2, p_3)$ ; let  $\mathbb{X}_i(\mathbf{p})$ ,  $i = 1, 2, 3$  be the corresponding point on the cross-ratio variety  $\mathfrak{X}$  and let

$$a = \arg(\mathbb{X}_1(\mathbf{p})), \quad b = \arg(\mathbb{X}_2(\mathbf{p})), \quad c = \arg(\mathbb{X}_3(\mathbf{p})).$$

There is an important note here: points of  $\mathbf{p}$  *cannot* all lie in the same (infinite)  $\mathbb{C}$ -circle. To see this, normalise so that

$$p_1 = (0, 0), \quad p_2 = (0, t), \quad p_3 = (0, s), \quad t, s \in \mathbb{R}.$$

Then conditions  $d(p_1, p_2) = d(p_1, p_3) = d(p_2, p_3)$  deduce

$$|t|^{1/2} = |s|^{1/2} = |t - s|^{1/2},$$

which cannot happen.

**Lemma 3.1.** *The triple  $P = (p_1, p_2, p_3)$  is equidistant if and only if*

$$|\mathbb{X}_1(\mathbf{p})| = |\mathbb{X}_2(\mathbf{p})| = 1.$$

*Proof.* Since  $P = (p_1, p_2, p_3)$  is an equidistant triple, we have

$$d(p_1, p_2) = d(p_2, p_3) = d(p_3, p_1),$$

where  $d$  is the Korányi distance. The result follows from the formulae

$$|\mathbb{X}_1(\mathbf{p})|^2 = \frac{d(p_2, p_1)}{d(p_3, p_1)}, \quad |\mathbb{X}_2(\mathbf{p})|^2 = \frac{d(p_3, p_2)}{d(p_3, p_1)}.$$

Note that the above imply as well  $|\mathbb{X}_3(\mathbf{p})| = 1$ .  $\square$

**Lemma 3.2.** *If  $P = (p_1, p_2, p_3)$  is an equidistant triple then  $a, b, c$  satisfy*

$$(3.1) \quad \cos a + \cos b + \cos c = \frac{3}{2}.$$

*Proof.* Consider the cross-ratio variety equations (2.1) and (2.2) as in the previous section. We may rewrite (2.2) equivalently as

$$(3.2) \quad 2|\mathbb{X}_1||\mathbb{X}_2| \cos c = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2|\mathbb{X}_1| \cos a - 2|\mathbb{X}_2| \cos b + 1.$$

If  $P$  is an equidistant triple then  $|\mathbb{X}_i(\mathbf{p})| = 1$ ,  $i = 1, 2, 3$  and thus (3.2) reduces to (3.1).  $\square$

**Lemma 3.3.** *If  $P = (p_1, p_2, p_3)$  is an equidistant triple,  $\mathbf{p} = (p_1, \infty, p_2, p_3)$  and  $(a, b, c)$  as above. If  $g \in G = \text{Sim}(\mathfrak{H})$ , we set*

$$P' = g(P) = (g(p_1), g(p_2), g(p_3)) = (p'_1, p'_2, p'_3), \quad \mathbf{p}' = (p'_1, \infty, p'_2, p'_3).$$

*Then*

$$a' = \arg(\mathbb{X}_1(\mathbf{p}')) = a, \quad b' = \arg(\mathbb{X}_2(\mathbf{p}')) = b, \quad c' = \arg(\mathbb{X}_3(\mathbf{p}')) = c.$$

*Proof.* The proof is immediate by invariance of cross-ratios.  $\square$

**Lemma 3.4.** *Let  $(a, b, c) \in [-2\pi/3, 2\pi/3]^3$  such that it satisfies*

$$\cos a + \cos b + \cos c = \frac{3}{2}.$$

*Then there exists an equidistant triple  $P = (p_1, p_2, p_3)$  such that if  $\mathbf{p} = (p_1, \infty, p_2, p_3)$  then*

$$\arg(\mathbb{X}_1(\mathbf{p})) = a, \quad \arg(\mathbb{X}_2(\mathbf{p})) = b, \quad \arg(\mathbb{X}_3(\mathbf{p})) = c.$$

*Proof.* Set  $2\eta = \arg(1 - e^{ia} - e^{ib})$ . We have

$$\begin{aligned} |1 - e^{ia} - e^{ib}|^2 &= 3 - 2\cos a - 2\cos b + 2\cos(a - b) \\ &= 3 - 4\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) + 4\cos^2\left(\frac{a-b}{2}\right) - 2 \\ &= 4\left(\cos\left(\frac{a-b}{2}\right) - \frac{1}{2}\cos\left(\frac{a+b}{2}\right)\right)^2 + \sin^2\left(\frac{a+b}{2}\right). \end{aligned}$$

This is strictly positive unless

$$(a, b, c) \in B = \{(\pi/3, -\pi/3, \pm\pi/3), (-\pi/3, \pi/3, \pm\pi/3)\}.$$

Assume first that  $(a, b, c) \notin B$  and set

$$A_1 = \frac{a - b - c}{2}, \quad A_4 = \frac{a - b + c}{2},$$

with  $A_1, A_4 \in (-\pi/2, \pi/2)$ . Notice that we have

$$\begin{aligned} 4\cos(A_1)\cos(A_4) &= 2\cos(a - b) + 2\cos c \\ &= 2\cos(a - b) + 3 - 2\cos a - 2\cos b \\ &= |1 - e^{ia} - e^{ib}|^2 > 0. \end{aligned}$$

Consider the triple  $P = (p_1, p_2, p_3)$  of points in  $\mathfrak{H}$  where

$$p_1 = \left( \sqrt{\cos(A_4)} e^{i(\frac{b-c}{2}-\eta)}, \sin(A_4) \right), \quad p_2 = (0, 0), \quad p_3 = \left( -\sqrt{\cos(A_1)} e^{i(\eta-\frac{a}{2})}, \sin(A_4) \right),$$

and the quadruple  $\mathbf{p} = (p_1, \infty, p_2, p_3)$ , with lifts:

$$\mathbf{p}_1 = \begin{bmatrix} -e^{-ia/2} \\ \sqrt{2\cos(A_4)} e^{-i\eta} \\ e^{i(c-b)/2} \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} e^{i(b+c)/2} \\ \sqrt{2\cos(A_1)} e^{i\eta} \\ -e^{ia/2} \end{bmatrix}.$$

Now,

$$\begin{aligned} \langle \mathbf{p}_1, \infty \rangle &= e^{i(c-b)/2}, & \langle \mathbf{p}_1, \mathbf{p}_2 \rangle &= -e^{-ia/2}, & \langle \infty, \mathbf{p}_2 \rangle &= 1, \\ \langle \infty, \mathbf{p}_3 \rangle &= -e^{-ia/2}, & \langle \mathbf{p}_2, \mathbf{p}_3 \rangle &= e^{-i(b+c)/2}, \end{aligned}$$

and

$$\begin{aligned} \langle \mathbf{p}_1, \mathbf{p}_3 \rangle &= e^{-ia} + e^{-ib} + \sqrt{2\cos(A_1)\cos(A_4)} \cdot e^{-2i\eta} \\ &= e^{-ia} + e^{-ib} + |1 - e^{-ia} - e^{-ib}| \cdot \frac{1 - e^{-ia} - e^{-ib}}{|1 - e^{-ia} - e^{-ib}|} \\ &= 1. \end{aligned}$$

This gives

$$\mathbb{X}_1 = e^{ia}, \quad \mathbb{X}_2 = e^{ib}, \quad \mathbb{X}_3 = e^{ic},$$

which proves our claim.

Finally, we consider  $(a, b, c) \in B$ ; we shall only treat the case where  $a = \pi/3$ ,  $b = -\pi/3$  and  $c = \pi/3$ , in other words when  $\mathbb{A}_4 = \pi/2$  and  $\mathbb{A}_1 = \pi/6$ . Then we set

$$\mathbf{p}_1 = \begin{bmatrix} \frac{-\sqrt{3}+i}{2} \\ 0 \\ \frac{1+i\sqrt{3}}{2} \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ 3^{1/4} \\ -\frac{\sqrt{3}+i}{2} \end{bmatrix}.$$

All other cases can be treated in a similar manner. □

**Lemma 3.5.** *Let  $(a, b, c) \in \mathcal{E}$  and consider from Lemma 3.4 the equidistant triple  $P = (p_1, p_2, p_3)$  of points in  $\mathfrak{H}$  which is such that*

$$a = \arg(\mathbb{X}_1(\mathbf{p})), \quad b = \arg(\mathbb{X}_2(\mathbf{p})), \quad c = \arg(\mathbb{X}_3(\mathbf{p})).$$

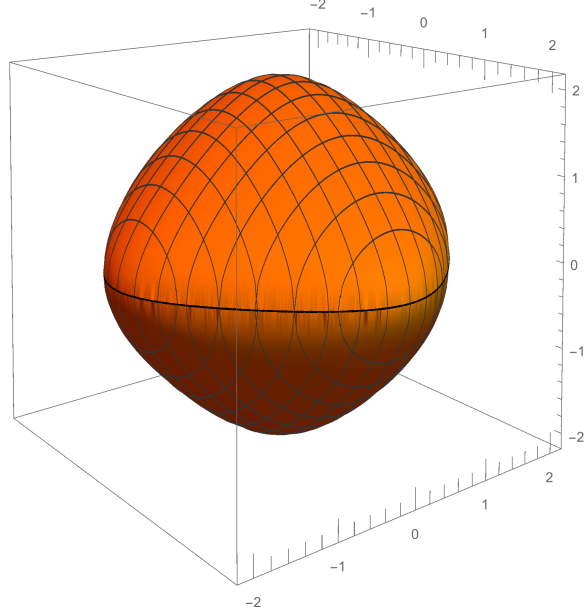
*Then any other equidistant triple  $P' = (p'_1, p'_2, p'_3)$  such that*

$$a = \arg(\mathbb{X}_1(\mathbf{p}')), \quad b = \arg(\mathbb{X}_2(\mathbf{p}')), \quad c = \arg(\mathbb{X}_3(\mathbf{p}'))$$

*is of the form  $P' = g(P)$ , where  $g \in G = \text{Sim}(\mathfrak{H})$ .*

*Proof.* We consider  $P$ ,  $P'$  and  $\mathbf{p}$ ,  $\mathbf{p}'$ , respectively. Since  $\mathbb{X}_i(\mathbf{p}) = \mathbb{X}_i(\mathbf{p}')$ ,  $i = 1, 2, 3$ , we have from Proposition 5.10 in [7] and Lemma 5.5 in [3] that since not all points in  $\mathbf{p}$  lie in the same  $\mathbb{C}$ -circle, there exists a  $g \in \text{PU}(2, 1)$  such that  $g(p_i) = p'_i$ ,  $i = 1, 2, 3$  and  $g(\infty) = \infty$ . That is,  $g \in G$  and the proof is complete. □

Central component of equidistant surface



**3.2. The Equidistant surface.** Equation (3.1) is the equation of a hypersurface in  $\mathbb{R}^3$  which we shall call *equidistant hypersurface* and denote it by  $\mathcal{E}$ . This hypersurface comprises of infinitely many connected components. Notice that we have

$$(a, b, c) \in \mathcal{E} \implies (a + 2n_1\pi, b + 2n_2\pi, c + 2n_3\pi) \in \mathcal{E}, \quad n_1, n_2, n_3 \in \mathbb{Z}.$$

On the other hand, since for instance

$$\cos a = \frac{3}{2} - \cos b - \cos c \geq -\frac{1}{2},$$

we have that  $\cos a$ , and in the same manner  $\cos b$  and  $\cos c$ , are  $\geq -1/2$ . We deduce that the connected components of  $\mathcal{E}$  may be taken by transporting the central component where

$$(a, b, c) \in [-2\pi/3, 2\pi/3]^3,$$

by multiples of  $2\pi$  in all possible directions, see the figure where the central component of  $\mathcal{E}$  is clearly shown.

**3.3. Proof of Theorem 1.1.** Given an equidistant triple  $P = (p_1, p_2, p_3)$  we consider the quadruple  $\mathbf{p} = (p_1, \infty, p_2, p_3)$  in the  $\text{PU}(2, 1)$ -configuration space of pairwise distinct points on the boundary  $\partial\mathbf{H}_{\mathbb{C}}^2$  of complex hyperbolic plane. Let  $\mathbb{X}_i(\mathbf{p})$ ,  $i = 1, 2, 3$  be the cross-ratios associated to  $\mathbf{p}$ ; the triple  $(\mathbb{X}_1(\mathbf{p}), \mathbb{X}_2(\mathbf{p}), \mathbb{X}_3(\mathbf{p}))$  defines a point in the cross-ratio variety  $\mathfrak{X}$ . In particular, in this case we have by Lemma 3.1 that  $|\mathbb{X}_i(\mathbf{p})| = 1$ ,  $i = 1, 2, 3$  and moreover, if

$$a = \arg(\mathbb{X}_1(\mathbf{p})), \quad b = \arg(\mathbb{X}_2(\mathbf{p})), \quad c = \arg(\mathbb{X}_3(\mathbf{p})),$$

then from Lemma 3.2

$$\cos a + \cos b + \cos c = \frac{3}{2}.$$

By Lemma 3.3 this equation is invariant by the diagonal action of  $G$  on the space of equidistant triples  $\mathcal{ET}$ : This proves that the map

$$\mathfrak{ET} \rightarrow \mathcal{E}, \quad [P] \mapsto (a, b, c),$$



is well-defined.

Conversely, if  $(a, b, c) \in \mathcal{E}$  where  $(a, b, c) \in [-2\pi/3, 2\pi/3]$ , by Lemma 3.4 there exists a  $P = (p_1, p_2, p_3) \in \mathcal{ET}$  such that if  $\mathbf{p} = (p_1, \infty, p_2, p_3)$  then  $a = \arg(\mathbb{X}_1(\mathbf{p}))$ ,  $b = \arg(\mathbb{X}_2(\mathbf{p}))$  and  $c = \arg(\mathbb{X}_3(\mathbf{p}))$ ; therefore  $\mathcal{ET} \rightarrow \mathcal{E}$  is onto. Finally, by Lemma 3.5  $\mathcal{ET} \rightarrow \mathcal{E}$  is 1–1 when the points in  $\mathbf{p}$  do not all lie in the same  $\mathbb{C}$ -circle and 2–1 when all points in  $\mathbf{p}$  lie in the same  $\mathbb{C}$ -circle. This concludes the proof of Theorem 1.1.  $\square$

**3.4. The  $\mathbb{C}$ -circle case.** The subset  $\mathcal{ET}_{\mathbb{C}}$  of  $\mathcal{ET}$  comprising equivalent equidistant triples of points lying on a  $\mathbb{C}$ -circle is of special interest. We can show in an elementary way that  $\mathcal{ET}_{\mathbb{C}}$  is just two points on the equidistant hypersurface  $\mathcal{E}$ . We start with a lemma:

**Lemma 3.6.** *With the assumptions of Section 3.1 let  $P = (p_1, p_2, p_3)$  and  $\mathbf{p} = (p_1, \infty, p_2, p_3)$ . Then*

$$a + b + c = -2\mathbb{A}_2 = -2\mathbb{A}(p_1, p_2, p_3) = -2\mathbb{A}.$$

*Proof.* We have

$$a + b + c = \mathbb{A}_1 - \mathbb{A}_2 - \mathbb{A}_2 - \mathbb{A}_4 + \mathbb{A}_4 - \mathbb{A}_1 = -2\mathbb{A}_2.$$

$\square$

We now prove

**Proposition 3.7.** *The subset  $\mathcal{ET}_{\mathbb{C}}$  of  $\mathcal{ET}$  comprising equivalence classes of equidistant triples of points lying on the same  $\mathbb{C}$ -circle is on bijection with the points  $(\pi/3, \pi/3, \pi/3)$  or  $(-\pi/3, -\pi/3, -\pi/3)$  of the equidistant surface  $\mathcal{E}$ .*

*Proof.* We will show that three equidistant points lie on a  $\mathbb{C}$ -circle if and only if the equidistant hypersurface reduces to the point  $(\pi/3, \pi/3, \pi/3)$  or  $(-\pi/3, -\pi/3, -\pi/3)$ . We start by assuming that the three points lie on a  $\mathbb{C}$ -circle, that is,  $\mathbb{A} = \pm\pi/2$ . Here,  $\mathbb{A} = \mathbb{A}(p_1, p_2, p_3)$ . Since from Lemma 3.6 we have

$$a + b + c = -2\mathbb{A},$$

Equation (3.1) becomes

$$\cos a + \cos b - \cos(a + b) = \frac{3}{2}.$$

This is written equivalently as

$$2(\cos a + \cos b) - 2\cos a \cos b + 2\sin a \sin b = \cos^2 a + \sin^2 a + \cos^2 b + \sin^2 b + 1,$$

or,

$$(\cos a + \cos b)^2 - 2(\cos a + \cos b) + 1 + (\sin a - \sin b)^2 = 0,$$

that is,

$$(\cos a + \cos b - 1)^2 + (\sin a - \sin b)^2 = 0.$$

Therefore we obtain,

$$\cos a + \cos b = 1 \quad \text{and} \quad \sin a = \sin b.$$

This gives

$$a = b = c = \pm \frac{\pi}{3}.$$

Conversely, suppose that three points lie on a  $\mathbb{C}$ -circle and  $\arg(\mathbb{X}_i) = \pm \frac{\pi}{3}$ . Then  $p_i$  are equidistant. Indeed, from equation (3.2) we have

$$|\mathbb{X}_1||\mathbb{X}_2| = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - |\mathbb{X}_1| - |\mathbb{X}_2| + 1.$$

Factoring out we may write equivalently

$$\left(|\mathbb{X}_1| - \frac{|\mathbb{X}_2|}{2} - \frac{1}{2}\right)^2 + \frac{3}{4}(|\mathbb{X}_2| - 1)^2 = 0.$$

This gives  $|\mathbb{X}_1| = |\mathbb{X}_2| = 1$  and therefore the points are equidistant.  $\square$

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